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LETTER TO THE EDITOR

Some remarks on free massive spin-3 field

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Abstract. The absence of local quadratic Lagrangian density of Takahashi–Umezawa type for the massive spin-3 tensor field has some important implications in both the relativistic and Euclidean field theories.

The recent result of Kawasaki *et al* (1975) and the author's calculation (Lim 1977, unpublished) have confirmed an earlier conjecture of the author that the general Lagrangian approach of Takahashi and Umezawa (1953) does not work for quantum fields with spin $s>3$ (Lim 1976). The basic idea is that the number of arbitrary parameters which appear in the Takahashi–Umezawa Lagrangian density increases rapidly as the spin value increases. For $s>2$ these parameters increase to such an extent that it is impossible to obtain a consistent set to guarantee the existence of a local Lagrangian density.

This is indeed the case for spin-3 massive tensor field $\phi_{\mu\nu\lambda}(x)$. If one assumes a second-order quadratic local Lagrangian density in the form

$$L=\tfrac{1}{2}\phi_{\mu\nu\lambda}(x)\Lambda^{\mu\nu\lambda,\rho\sigma\kappa}(\partial)\phi_{\rho\sigma\kappa}(x), \tag{1}$$

where $\Lambda^{\mu\nu\lambda,\rho\sigma\kappa}(\partial)$ is a local second-order differential operator, then the Euler–Lagrange equation

$$\Lambda^{\mu\nu\lambda,\rho\sigma\kappa}(\partial)\phi_{\rho\sigma\kappa}(x)=0 \tag{2}$$

so obtained cannot be decomposed into the Klein–Gordon equation and all the subsidiary conditions, as required by Takahashi–Umezawa formulation.

This result also confirms the belief that auxiliary fields need to be introduced in order to obtain a consistent Lagrangian field theory for spin $s>2$ (Chang 1967, Singh and Hagen 1974, Kobayashi and Mori 1975). Auxiliary fields are required to eliminate the non-local terms involving $(\partial^2)^{-1}$ (i.e. the inverse of D'Alembertian operator) in the Lagrangian density. In other words, $\Lambda(\partial)$ in (1) is not a *bona fide* local operator and one gets a non-local Lagrangian field theory for massive spin-3 particle. This does not come as a surprise since the auxiliary field method has been frequently employed to obtain a consistent quantum theory of higher spin fields with interaction.

Since the relativistic propagator is given by the negative inverse of the Fourier transform of $\Lambda(\partial)$, so the non-existence of a local $\Lambda(\partial)$ for the massive spin-3 tensor field implies that the corresponding Euclidean propagator does not have a local inverse. This means that the generalised Gaussian random field constructed with this Euclidean propagator as covariance does not satisfy the Markov property of Nelson (1973a, b), since the necessary condition for a generalised Gaussian field to be

Markovian is that the inverse of the covariance be local (Nelson 1973a, b, Goodman 1975). Thus the method we used to construct Euclidean massive Markov fields with spin $s \leq 2$ fails for the spin-3 case (Lim 1976).

Another alternative of getting a massive spin-3 theory is to use the $(2s+1)$-component field of Weinberg (1964). However such an approach does not require the explicit construction of the Lagrangian density, so the Markov property of the corresponding Euclidean field is not transparent (Ozkaynak 1974).

However, one can still construct a Euclidean Gaussian field from the relativistic spin-3 field. For a massive tensor field $\phi^{\mu\nu\lambda}(x)$ which is traceless, divergenceless and totally symmetric, the two-point Wightman function is given by

$$\begin{aligned} W^{\mu\nu\lambda,\mu'\nu'\lambda'}&(x-x') \\ &= \langle \phi^{\mu\nu\lambda}(x)\phi^{\mu'\nu'\lambda'}(x')\rangle \\ &= \operatorname*{sym}_{(\mu\nu\lambda),(\mu'\nu'\lambda')} (\mathrm{d}^{\mu\mu'}\mathrm{d}^{\nu\nu'}\mathrm{d}^{\lambda\lambda'} - \tfrac{3}{5}\mathrm{d}^{\mu\nu}\mathrm{d}^{\mu'\nu'}\mathrm{d}^{\lambda\lambda'})W(x-x'), \end{aligned} \tag{4}$$

where $W(x-x')$ is the two-point Wightman function for the corresponding massive scalar field, and $\mathrm{d}^{\mu\nu} = g^{\mu\nu} + m^{-2}\partial^\mu\partial^\nu$†, sym$(\mu\nu\lambda)$, $(\mu'\nu'\lambda')$ denotes symmetrisation in μ, ν, λ and μ', ν', λ'.

To obtain the corresponding Euclidean (or Schwinger) two-point function one needs to replace all the Minkowski metric $g^{\mu\nu}$ by the Euclidean metric δ_{ij}, in addition to the usual analytic continuation to pure imaginary time of the Wightman two-point function. This can be achieved by the following transformation:

$$S^1_{ijk,i'j'k'}(x-x') = A_{i\mu}A_{j\nu}A_{k\lambda}A_{i'\mu'}A_{j'\nu'}A_{k'\lambda'}W^{\mu\nu\lambda,\mu'\nu'\lambda'}(\boldsymbol{x}-\boldsymbol{x}', \mathrm{i}(x_0-x_0')), \tag{5}$$

where $A_{i\mu} = 1$ if $i = \mu = 1, 2, 3$; $A_{40} = i$ and $A_{i\mu} = 0$ otherwise.

The Euclidean two-point function given in equation (5) is not positive semi-definite. Note that we have changed all the $g^{\mu\nu}$ to δ_{ij} without altering the relevant numerical coefficients, hence the trace of $S^1_{ijk,i'j'k'}$ is non-zero. However, one can obtain the correct two-point Schwinger function by replacing equation (5) by

$$S^0_{ijk,i'j'k'}(x-x') = A_{ijk,i'j'k';\mu\nu\lambda,\mu'\nu'\lambda'}W^{\mu\nu\lambda,\mu'\nu'\lambda'}(\boldsymbol{x}-\boldsymbol{x}', \mathrm{i}(x_0-x_0')), \tag{6a}$$

where

$$A_{ijk,i'j'k';\mu\nu\lambda,\mu'\nu'\lambda'} = \tfrac{1}{18}\sum_c (A_{i\mu}A_{j\nu} + \tfrac{1}{4}\delta_{ij}g_{\mu\nu})(A_{i'\mu'}A_{j'\nu'} + \tfrac{1}{4}\delta_{i'j'}g_{\mu'\nu'})(A_{k\lambda}A_{k'\lambda'}), \tag{6b}$$

with Σ_c denoting the sum over all distinct combinations of indices $(i\mu, j\nu, k\lambda)$ and $(i'\mu', j'\nu', k'\lambda')$. Since $\phi^{\mu\lambda}_\mu(x) = 0$, the additional term $\frac{1}{4}\delta_{ij}g_{\mu\nu}$ in (6) adds only contact terms to the two-point function, containing delta function and its derivatives (recall that Schwinger functions are defined from Wightman functions but for distributions concentrated at coinciding points). It can be checked that $S^0_{ijk,i'j'k'}$ is traceless and positive semi-definite.

One can now construct the Euclidean one-particle space K for the massive spin-3 particle following the usual method with real Schwartz symmetric tensor test functions as elements and the two-point Schwinger function (6) as the inner product. The Euclidean massive spin-3 field θ_{ijk} can then be defined as the generalised Gaussian random field over K with mean zero and covariance given by the inner product in K. Clearly, such a field is covariant under the full Euclidean group ISO(4). It is also not

† Here we have used the convention $g^{0\mu} = +\delta^{0\mu}$, $g^{ij} = -\delta^{ij}$.

difficult to see that θ_{ijk} is reflexive. Let τ be a unitary representation of ISO(4) on the underlying probability space such that

$$\tau(R,a)\theta_{ijk}(f)\tau^{-1}(R,a)=\sum_{i',j',k'} R_{ii'}^{-1}R_{jj'}^{-1}R_{kk'}^{-1}\theta_{i'j'k'}(f_{R,a}), \tag{7}$$

where $R\in \mathrm{SO}(4)$, $a\in\mathbb{R}^4$ and $f_{R,a}(x)=f(R^{-1}(x-a))$. If $\tau(\rho)$ is the reflection in the hyperplane $x_4=0$ (denoted by Π_0), then for all $f\in K$ with support in Π_0 we have

$$\tau(\rho)\theta_{ijk}(f)\tau^{-1}(\rho)=(-1)^{\delta_{i4}+\delta_{j4}+\delta_{k4}}\theta_{ijk}(f_\rho)=\theta_{ijk}(f), \tag{8}$$

where $f_\rho=f(\boldsymbol{x},-x_4)$. Equation (8) guarantees the reflexivity of θ_{ijk} (see Lim 1976 for detailed argument).

Finally we note that θ_{ijk} constructed above is non-Markovian. This is due to the fact that $S^0_{ijk,i'j'k'}$ does not have a local inverse. We remark that the matrix transformation in (6) does not alter the non-locality of the inverse of $W^{\mu\nu\lambda,\mu'\nu'\lambda'}$ since $A_{ijk,i'j'k';\mu\nu\lambda,\mu'\nu'\lambda'}$ is just a non-singular constant matrix.

Thus we have constructed a rank-3 Euclidean Gaussian tensor field for spin-3 massive particle. Even though it is non-Markovian, it satisfies the axioms Osterwalder and Schrader (1973, 1975) and so it can be put into this more general framework. The above construction can be easily generalised to spin $s>3$ tensor fields, which most likely are non-Markovian.

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